13-3 Sum and Difference of Angles Identities

Find the exact value of each expression.

1. $\cos 165°$

ANSWER:

$-\dfrac{\sqrt{2}+\sqrt{6}}{4}$

2. $\cos 105°$

ANSWER:

$\dfrac{\sqrt{2}-\sqrt{6}}{4}$

3. $\cos 75°$

ANSWER:

$\dfrac{\sqrt{6}-\sqrt{2}}{4}$

4. $\sin(-30°)$

ANSWER:

$-\dfrac{1}{2}$

5. $\sin 135°$

ANSWER:

$\dfrac{\sqrt{2}}{2}$

6. $\sin(-210°)$

ANSWER:

$\dfrac{1}{2}$

7. **CCSS MODELING** Refer to the beginning of the lesson. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude. The first signal can be modeled by the equation $y = 20\sin(3\theta + 45°)$. The second signal can be modeled by the equation $y = 20\sin(3\theta + 225°)$

a. Find the sum of the two functions.
b. What type of interference results when signals modeled by the two equations are combined?

ANSWER:

a. 0
b. The interference is destructive. The signals cancel each other completely.

Verify that each equation is an identity.

8. $\sin(90° + \theta) = \cos\theta$

ANSWER:

$$\sin(90° + \theta) \overset{?}{=} \cos\theta$$
$$\sin 90° \cos\theta + \cos 90° \sin\theta \overset{?}{=} \cos\theta$$
$$1 \cdot \cos\theta + 0 \cdot \sin\theta \overset{?}{=} \cos\theta$$
$$\cos\theta = \cos\theta\checkmark$$

9. $\cos\left(\dfrac{3\pi}{2} - \theta\right) = -\sin\theta$

ANSWER:

$$\cos\left(\dfrac{3\pi}{2} - \theta\right) \overset{?}{=} -\sin\theta$$
$$\cos\dfrac{3\pi}{2}\cos\theta + \sin\dfrac{3\pi}{2}\sin\theta \overset{?}{=} -\sin\theta$$
$$0 \cdot \cos\theta - 1 \cdot \sin\theta \overset{?}{=} -\sin\theta$$
$$-\sin\theta = -\sin\theta\checkmark$$

13-3 Sum and Difference of Angles Identities

10. $\tan\left(\theta+\dfrac{\pi}{2}\right)=-\cot\theta$

ANSWER:

$$\tan\left(\theta+\frac{\pi}{2}\right)=-\cot\theta$$

$$\frac{\sin\left(\theta+\dfrac{\pi}{2}\right)}{\cos\left(\theta+\dfrac{\pi}{2}\right)}\overset{?}{=}-\cot\theta$$

$$\frac{\sin\theta\cos\dfrac{\pi}{2}+\cos\theta\sin\dfrac{\pi}{2}}{\cos\theta\cos\dfrac{\pi}{2}-\sin\theta\sin\dfrac{\pi}{2}}\overset{?}{=}-\cot\theta$$

$$\frac{(\sin\theta)\cdot0+(\cos\theta)\cdot1}{(\cos\theta)\cdot0-(\sin\theta)\cdot1}\overset{?}{=}-\cot\theta$$

$$-\frac{\cos\theta}{\sin\theta}\overset{?}{=}-\cot\theta$$

$$-\cot\theta=-\cot\theta\checkmark$$

11. $\sin(\theta+\pi)=-\sin\theta$

ANSWER:

$$\sin(\theta+\pi)\overset{?}{=}-\sin\theta$$

$$\sin\theta\cos\pi+\cos\theta\sin\pi\overset{?}{=}-\sin\theta$$

$$(\sin\theta)(-1)+(\cos\theta)(0)\overset{?}{=}-\sin\theta$$

$$-\sin\theta=-\sin\theta\checkmark$$

Find the exact value of each expression.

12. $\sin165°$

ANSWER:

$$\frac{\sqrt{6}-\sqrt{2}}{4}$$

13. $\cos135°$

ANSWER:

$$-\frac{\sqrt{2}}{2}$$

14. $\cos\dfrac{7\pi}{12}$

ANSWER:

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

15. $\sin\dfrac{\pi}{12}$

ANSWER:

$$\frac{\sqrt{6}-\sqrt{2}}{4}$$

16. $\tan195°$

ANSWER:

$$2-\sqrt{3}$$

17. $\cos\left(-\dfrac{\pi}{12}\right)$

ANSWER:

$$\frac{\sqrt{2}+\sqrt{6}}{4}$$

18. **ELECTRONICS** In a certain circuit carrying alternating current, the formula $c = 2\sin(120t)$ can be used to find the current c in amperes after t seconds.
a. Rewrite the formula using the sum of two angles.
b. Use the sum of angles formula to find the exact current at $t = 1$ second.

ANSWER:

a. $c = 2\sin(90t + 30t)$

b. $\sqrt{3}$ amperes

Verify that each equation is an identity.

19. $\cos\left(\dfrac{\pi}{2}+\theta\right)=-\sin\theta$

ANSWER:

$$\cos\left(\dfrac{\pi}{2}+\theta\right)\overset{?}{=}-\sin\theta$$

$$\cos\dfrac{\pi}{2}\cos\theta-\sin\dfrac{\pi}{2}\sin\theta\overset{?}{=}-\sin\theta$$

$$0\cdot\cos\theta-1\cdot\sin\theta\overset{?}{=}-\sin\theta$$

$$-\sin\theta=-\sin\theta\checkmark$$

20. $\cos(60°+\theta)=\sin(30°-\theta)$

ANSWER:

$$\cos(60°+\theta)\overset{?}{=}\sin(30°-\theta)$$

$$\cos60°\cos\theta-\sin60°\sin\theta\overset{?}{=}\sin30°\cos\theta-\cos30°\sin\theta$$

$$\dfrac{1}{2}\cos\theta-\dfrac{\sqrt{3}}{2}\sin\theta=\dfrac{1}{2}\cos\theta-\dfrac{\sqrt{3}}{2}\sin\theta\checkmark$$

21. $\cos(180°+\theta)=-\cos\theta$

ANSWER:

$$\cos(180°+\theta)\overset{?}{=}-\cos\theta$$

$$\cos180°\cos\theta-\sin180°\sin\theta\overset{?}{=}-\cos\theta$$

$$-1\cdot\cos\theta-0\cdot\sin\theta\overset{?}{=}-\cos\theta$$

$$-\cos\theta=-\cos\theta\checkmark$$

22. $\tan(\theta+45°)=\dfrac{1+\tan\theta}{1-\tan\theta}$

ANSWER:

$$\tan(\theta+45°)\overset{?}{=}\dfrac{1+\tan\theta}{1-\tan\theta}$$

$$\dfrac{\tan\theta+\tan45°}{1-\tan\theta\tan45°}\overset{?}{=}\dfrac{1+\tan\theta}{1-\tan\theta}$$

$$\dfrac{\tan\theta+1}{1-(\tan\theta)(1)}\overset{?}{=}\dfrac{1+\tan\theta}{1-\tan\theta}$$

$$\dfrac{1+\tan\theta}{1-\tan\theta}=\dfrac{1+\tan\theta}{1-\tan\theta}\checkmark$$

23. **CCSS REASONING** The monthly high temperatures for Minneapolis, Minnesota, can be modeled by the

equation $y=31.65\sin\left(\dfrac{\pi}{6}x-2.09\right)+52.35$, where

the months x are represented by January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by the

equation $y=30.15\sin\left(\dfrac{\pi}{6}x-2.09\right)+32.95$.

a. Write a new function by adding the expressions on the right side of each equation and dividing the result by 2.

b. What is the meaning of the function you wrote in part a?

ANSWER:

a. $y=30.9\sin\left(\dfrac{\pi}{6}x-2.09\right)+42.65$

b. The new function represents the average of the high and low temperatures for each month.

Find the exact value of each expression.

24. $\tan165°$

ANSWER:

$-2+\sqrt{3}$

25. $\sec1275°$

ANSWER:

$\sqrt{2}-\sqrt{6}$

26. $\sin735°$

ANSWER:

$\dfrac{\sqrt{6}-\sqrt{2}}{4}$

27. $\tan\dfrac{23\pi}{12}$

ANSWER:

$-2+\sqrt{3}$

28. $\csc\dfrac{5\pi}{12}$

    ANSWER:

    $\sqrt{6}-\sqrt{2}$

29. $\cot\dfrac{113\pi}{12}$

    ANSWER:

    $2-\sqrt{3}$

30. **FORCE** In the figure given below, the effort F necessary to hold a safe in position non a ramp is given by $F=\dfrac{W\left(\sin A+\mu\cos A\right)}{\cos A-\mu\sin A}$, where W is the weight of the safe and $\mu=\tan\theta$. Show that $F=W\tan(A+\theta)$.
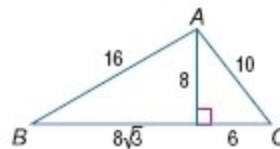

    ANSWER:

    $$F=\frac{W\left(\sin A+\mu\cos A\right)}{\cos A-\mu\sin A}$$
    $$=\frac{W\left(\sin A+\tan\theta\cos A\right)}{\cos A-\tan\theta\sin A}$$
    $$=\frac{W\left(\dfrac{\sin A}{\cos A}+\dfrac{\tan\theta\cos A}{\cos A}\right)}{\dfrac{\cos A}{\cos A}-\dfrac{\tan\theta\sin A}{\cos A}}$$
    $$=\frac{W\left(\tan A+\tan\theta\right)}{1-\tan A\tan\theta}$$
    $$=W\tan\left(A+\theta\right)$$

31. **QUILTING** As part of a quilt that is being made, the quilter places two right triangular swatches together to make a new triangular piece. One swatch has sides 6 inches, 8 inches, and 10 inches long. The other swatch has sides 8 inches, $8\sqrt{3}$ inches, and 16 inches long. The pieces are placed with the sides of eight inches against each other, as shown I the figure, to form triangle ABC.

    a. What is the exact value of the sine of angle BAC?

    b. What is the exact value of the cosine of angle BAC?

    c. What is the measure of angle BAC?

    d. Is the new triangle formed from the two triangles also a right triangle?

    ANSWER:

    a. $\dfrac{3+4\sqrt{3}}{10}$

    b. $\dfrac{4-3\sqrt{3}}{10}$

    c. 96.9°

    d. no

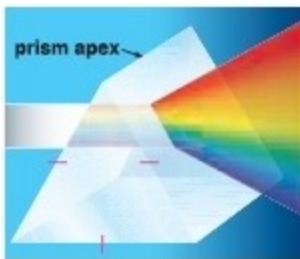32. **OPTICS** When light passes symmetrically through a prism, the index of refraction n of the glass with respect to air is $n = \dfrac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin\frac{b}{2}}$, where a is the measure of the deviation angle and b is the measure of the prism apex angle.

a. Show that for the prism shown,

$n = \sqrt{3}\sin\dfrac{a}{2} + \cos\dfrac{a}{2}$

b. Find n for the prism shown.

prism apex

ANSWER:

a.

$$\dfrac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin\frac{b}{2}} = \dfrac{\sin\left[\frac{1}{2}(a+60^\circ)\right]}{\sin\frac{60^\circ}{2}}$$

$$= \dfrac{\sin\left(\frac{a}{2}+30^\circ\right)}{\sin 30^\circ}$$

$$= \dfrac{\sin\frac{a}{2}\cos 30^\circ + \cos\frac{a}{2}\sin 30^\circ}{\sin 30^\circ}$$

$$= \dfrac{\left(\sin\frac{a}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\cos\frac{a}{2}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}}$$

$$= \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2}$$

b. $\sqrt{3}$

33. **MULTIPLE REPRESENTATIONS** In this problem, you will disprove the hypothesis that $\sin(A+B) = \sin A + \sin B$.

a. TABULAR Complete the table

A	B	sin A	sin B	sin (A + B)	sin A + sin B
30°	90°				
45°	60°				
60°	45°				
90°	30°				

b. GRAPHICAL Assume that B is always 15° less than A. Use a graphing calculator to graph $\sin(x+x-15)$ and $\sin x + \sin(x-15)$ on the same screen.
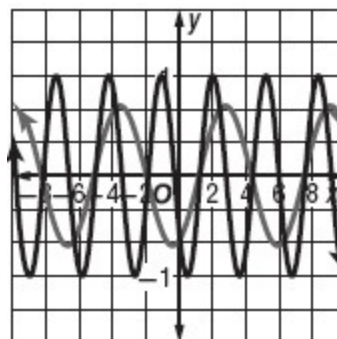
c. ANALYTICAL Determine whether $\cos(A+B) = \cos A + \cos B$ is an identity. Explain your reasoning.

ANSWER:

a.

A	B	sin A	sin B	sin (A + B)	sin A + sin B
30°	90°	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{3}{2}$
45°	60°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{2}+\sqrt{3}}{2}$
60°	45°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{2}+\sqrt{3}}{2}$
90°	30°	1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{2}$

b.

c. No; a counterexample is $\cos(30^\circ + 45^\circ) = \cos 30^\circ + \cos 45^\circ$, which equals $\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2}$ or about 1.5731. Since a cosine value cannot be greater than 1, this statement must be false.

13-3 Sum and Difference of Angles Identities

Verify that each equation is an identity.

34. $\sin(A+B) = \dfrac{\tan A + \tan B}{\sec A \sec B}$

ANSWER:

$$\sin(A+B) \overset{?}{=} \dfrac{\tan A + \tan B}{\sec A \sec B}$$

$$\sin(A+B) \overset{?}{=} \dfrac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}}$$

$$\sin(A+B) \overset{?}{=} \dfrac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}} \cdot \dfrac{\cos A \cos B}{\cos A \cos B}$$

$$\sin(A+B) \overset{?}{=} \dfrac{\sin A \cos B + \cos A \sin B}{1}$$

$$\sin(A+B) = \sin(A+B) \checkmark$$

35. $\cos(A+B) = \dfrac{1 - \tan A \tan B}{\sec A \sec B}$

ANSWER:

$$\cos(A+B) \overset{?}{=} \dfrac{1 - \tan A \tan B}{\sec A \sec B}$$

$$\cos(A+B) \overset{?}{=} \dfrac{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}}$$

$$\cos(A+B) \overset{?}{=} \dfrac{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}} \cdot \dfrac{\cos A \cos B}{\cos A \cos B}$$

$$\cos(A+B) \overset{?}{=} \dfrac{\cos A \cos B - \sin A \sin B}{1}$$

$$\cos(A+B) = \cos(A+B) \checkmark$$

36. $\sec(A-B) = \dfrac{\sec A \sec B}{1 + \tan A \tan B}$

ANSWER:

$$\sec(A-B) \overset{?}{=} \dfrac{\sec A \sec B}{1 + \tan A \tan B}$$

$$\sec(A-B) \overset{?}{=} \dfrac{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}}{1 + \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

$$\sec(A-B) \overset{?}{=} \dfrac{\dfrac{1}{\cos A} \cdot \dfrac{1}{\cos B}}{1 + \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}} \cdot \dfrac{\cos A \cos B}{\cos A \cos B}$$

$$\sec(A-B) \overset{?}{=} \dfrac{1}{\cos A \cos B + \sin A \sin B}$$

$$\sec(A-B) \overset{?}{=} \dfrac{1}{\cos(A-B)}$$

$$\sec(A-B) = \sec(A-B) \checkmark$$

37. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

ANSWER:

$$\sin(A+B)\sin(A-B) \overset{?}{=} \sin^2 A - \sin^2 B$$

$$(\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \overset{?}{=} \sin^2 A - \sin^2 B$$

$$(\sin A \cos B)^2 - (\cos A \sin B)^2 \overset{?}{=} \sin^2 A - \sin^2 B$$

$$\sin^2 A \cos^2 B - \cos^2 A \sin^2 B \overset{?}{=} \sin^2 A - \sin^2 B$$

$$\sin^2 A \cos^2 B + \sin^2 A \sin^2 B - \sin^2 A \sin^2 B - \cos^2 A \sin^2 B \overset{?}{=} \sin^2 A - \sin^2 B$$

$$\sin^2 A(\cos^2 B + \sin^2 B) - \sin^2 B(\sin^2 A + \cos^2 A) \overset{?}{=} \sin^2 A - \sin^2 B$$

$$(\sin^2 A)(1) - (\sin^2 B)(1) \overset{?}{=} \sin^2 A - \sin^2 B$$

$$\sin^2 A - \sin^2 B = \sin^2 A - \sin^2 B \checkmark$$

38. **REASONING** Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\dfrac{\pi}{3} - \theta\right)\cos\left(\dfrac{\pi}{3} + \theta\right) - \cos\left(\dfrac{\pi}{3} - \theta\right)\sin\left(\dfrac{\pi}{3} + \theta\right)$$

ANSWER:

$\sin(-2\theta)$

39. **WRITING IN MATH** Use the information at the beginning of the lesson and in Exercise 7 to explain how the sum and difference identities are used to describe wireless Internet interference. Include an explanation of the difference between constructive and destructive interference.
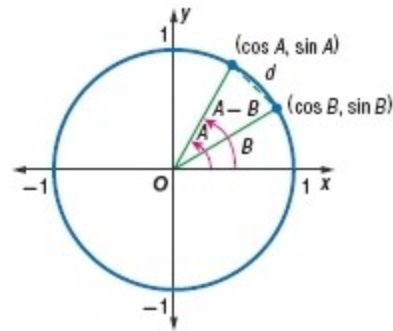
ANSWER:

Sample answer: To determine wireless Internet Interference, you need to determine the sine or cosine of the sum or difference of two angles. Interference occurs when waves pass through the same space at the same time. When the combined waves have a greater amplitude, constructive interference results. When the combined waves have a smaller amplitude, destructive interference results.

40. **CHALLENGE** Derive an identity for $\cot (A + B)$ in terms of $\cot A$ and $\cot B$.
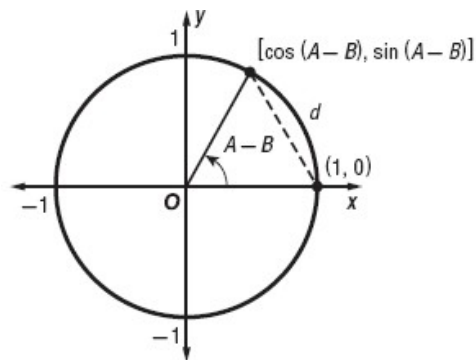
ANSWER:

$$\cot(A+B) = \frac{1}{\tan(A+B)}$$

$$= \frac{1}{\frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$= \frac{1 - \tan A \tan B}{\tan A + \tan B}$$

$$= \frac{1 - \frac{1}{\cot A} \cdot \frac{1}{\cot B}}{\frac{1}{\cot A} + \frac{1}{\cot B}} \cdot \frac{\cot A \cot B}{\cot A \cot B}$$

$$= \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

41. **CCSS ARGUMENTS** The figure shown has two angles A and B in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos B, \sin B)$ and $(x_2, y_2) = (\cos A, \sin A)$.

ANSWER:

$$d = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$
$$d^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$d^2 = (\cos^2 A - 2\cos A\cos B + \cos^2 B) + (\sin^2 A - 2\sin A\sin B + \sin^2 B)$$
$$d^2 = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B$$
$$d^2 = 1 + 1 - 2\cos A\cos B - 2\sin A\sin B \left(\begin{array}{c}\text{since, } \sin^2 A + \cos^2 A = 1 \\ \text{and } \sin^2 B + \cos^2 B = 1\end{array}\right)$$
$$d^2 = 2 - 2\cos A\cos B - 2\sin A\sin B$$

Now find the value of d^2 when the angle having measure $A - B$ is in standard position on the unit circle, as shown in the figure at the left.

$$d = \sqrt{[\cos(A-B)-1]^2 + [\sin(A-B)-0]^2}$$
$$d^2 = [\cos(A-B)-1]^2 + [\sin(A-B)-0]^2$$
$$d^2 = [\cos^2(A-B)-2\cos(A-B)+1] + \sin^2(A-B)$$
$$d^2 = \cos^2(A-B) + \sin^2(A-B) - 2\cos(A-B)+1$$
$$d^2 = 1 - 2\cos(A-B)+1$$
$$d^2 = 2 - 2\cos(A-B)$$

42. **OPEN ENDED** Consider the following theorem. *If A, B, and C are the angles of an oblique triangle, then tan A + tan B + tan C = tan A tan B tan C. Choose values for A, B, and C. Verify that the conclusion is true for your specific values.*

ANSWER:

Sample answer :

$A = 35°, B = 60°, C = 85°;$

$$0.7002 + 1.7321 + 11.4301 \overset{?}{=} (0.7002) \cdot (1.7321)(11.4301)$$
$$13.86 = 13.86\checkmark$$

43. **GRIDDED RESPONSE** The mean of seven numbers is 0. The sum of three of the numbers is –9. What is the sum of the remaining four numbers?

ANSWER:

9

44. The variables *a, b, c, d,* and *f* are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of *f*?

A 74
B 144
C 146
D 256

ANSWER:

C

45. **SAT/ACT** Solve $x^2 - 5x < 14$.

F $\{x| -7 < x < 2\}$
G $\{x|x < -7 \text{ or } x > 2\}$
H $\{x| -2 < x < 7\}$
J $\{x|x < -2 \text{ or } x > 7\}$
K $\{x|x > -2 \text{ and } x < 7\}$

ANSWER:

H

46. **PROBABILITY** A math teacher is randomly distributing 15 yellow pencils and 10 green pencils. What is the probability that the first pencil she hands out will be yellow and the second pencil will be green?

A $\dfrac{1}{24}$

B $\dfrac{1}{4}$

C $\dfrac{2}{5}$

D $\dfrac{23}{25}$

ANSWER:

B

Verify that each equation is an identity.

47. $\dfrac{\sin\theta}{\tan\theta} + \dfrac{\cos\theta}{\cot\theta} = \cos\theta + \sin\theta$

ANSWER:

$$\dfrac{\sin\theta}{\tan\theta} + \dfrac{\cos\theta}{\cot\theta} \overset{?}{=} \cos\theta + \sin\theta$$

$$\dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \dfrac{\cos\theta}{\frac{\cos\theta}{\sin\theta}} \overset{?}{=} \cos\theta + \sin\theta$$

$$\sin\theta \cdot \dfrac{\cos\theta}{\sin\theta} + \cos\theta \cdot \dfrac{\sin\theta}{\cos\theta} \overset{?}{=} \cos\theta + \sin\theta$$

$$\cos\theta + \sin\theta = \cos\theta + \sin\theta\checkmark$$

48. $\sec\theta(\sec\theta - \cos\theta) = \tan^2\theta$

ANSWER:

$$\sec\theta(\sec\theta - \cos\theta) \overset{?}{=} \tan^2\theta$$

$$\dfrac{1}{\cos\theta}\left(\dfrac{1}{\cos\theta} - \cos\theta\right) \overset{?}{=} \tan^2\theta$$

$$\dfrac{1}{\cos^2\theta} - 1 \overset{?}{=} \tan^2\theta$$

$$\sec^2\theta - 1 \overset{?}{=} \tan^2\theta$$

$$\tan^2\theta = \tan^2\theta\checkmark$$

Simplify each expression.

49. $\sin\theta\csc\theta - \cos^2\theta$

    ANSWER:

    $\sin^2\theta$

50. $\cos^2\theta\sec\theta\csc\theta$

    ANSWER:

    $\cot\theta$

51. $\cos\theta + \sin\theta\tan\theta$

    ANSWER:

    $\sec\theta$

52. **GUITAR** When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz).
    a. Find the period of this function.
    b. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit, and let the minimum distance below this position have a value of 1 unit.

    ANSWER:

    a. $\dfrac{1}{440}$ second

    b.
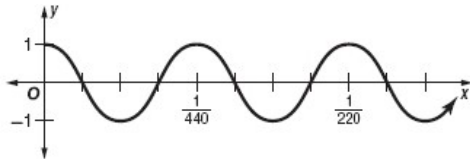
Prove that each statement is true for all positive integers.

53. $4^n - 1$ is divisible by 3.

    ANSWER:

    Step 1: $4^1 - 1 = 3$, which is divisible by 3. The statement is true for $n = 1$.

    Step 2: Assume that $4^k - 1$ is divisible by 3 for some positive integer k. This means that $4^k - 1 = 3r$ for some whole number r.

    Step 3:

    $$4^k - 1 = 3r$$
    $$4^k = 3r + 1$$
    $$4^{k+1} = 12r + 4$$
    $$4^{k+1} - 1 = 12r + 3$$
    $$4^{k+1} - 1 = 3(4r + 1)$$

    Since r is a whole number, $4r + 1$ is a whole number. Thus, $4^{k+1} - 1$ is divisible by 3, so the statement is true for $n = k + 1$. Therefore, $4^n - 1$ is divisible by 3 for all positive integers n.. Therefore, $4^n - 1$ is divisible by 3 for all positive integers n.

54. $5^{n} + 3$ is divisible by 4.

   ANSWER:

   Step 1: $5^{1} + 3 = 8$, which is divisible by 4. The statement is true for $n = 1$.

   Step 2: Assume that $5^{k} + 3$ is divisible by 4 for some positive integer k. This means that $5^{k} + 3 = 4r$ for some whole number r.

   Step 3:

   $$5^{k} + 3 = 4r$$
   $$5^{k} = 4r - 3$$
   $$5^{k+1} = 20r - 15$$
   $$5^{k+1} + 3 = 20r - 12$$
   $$5^{k+1} + 3 = 4(5r - 3)$$

   Since r is a positive integer, $5r - 3$ is a positive integer. Thus, $5^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$. Therefore, $5^{n} + 3$ is divisible by 4 for all positive integers n.

   Solve each equation.

55. $7 + \sqrt{4x + 8} = 9$

   ANSWER:
   -1

56. $\sqrt{y + 21} - 1 = \sqrt{y + 12}$

   ANSWER:
   4

57. $\sqrt{4z + 1} = 3 + \sqrt{4z - 2}$

   ANSWER:
   no solution