13-5 Solving Trigonometric Equations

CCSS REGULARITY Solve each equation if $0° \le \theta \le 360°$.

1. $2\sin\theta + 1 = 0$

ANSWER:

$210°, 330°$

2. $\cos^2\theta + 2\cos\theta + 1 = 0$

ANSWER:

$180°$

3. $\cos 2\theta + \cos\theta = 0$

ANSWER:

$60°, 180°, 300°$

4. $2\cos\theta = 1$

ANSWER:

$60°, 300°$

5. $\cos\theta = -\frac{\sqrt{3}}{2}$

ANSWER:

$150°, 210°$

6. $\sin 2\theta = -\frac{\sqrt{3}}{2}$

ANSWER:

$120°, 150°, 300°, 330°$

7. $\cos 2\theta = 8 - 15\sin\theta$

ANSWER:

$30°, 150°$

8. $\sin\theta + \cos\theta = 1$

ANSWER:

$0°,\ 90°,\ 360°$

Solve each equation for all values of θ if θ is measured in radians.

9. $4\sin^2\theta - 1 = 0$

ANSWER:

$\pm\frac{\pi}{6} + 2k\pi \text{ or } \pm\frac{5\pi}{6} + 2k\pi$

10. $2\cos^2\theta = 1$

ANSWER:

$\frac{\pi}{4} + \frac{k}{2}\pi$

11. $\cos 2\theta \sin\theta = 1$

ANSWER:

$\frac{3\pi}{2} + 2k\pi$

12. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{2}$

ANSWER:

$\frac{\pi}{2} + 4\pi k$

13. $\cos 2\theta + 4\cos\theta = -3$

ANSWER:

$\pi + 2k\pi$

14. $\sin\frac{\theta}{2} + \cos\theta = 1$

ANSWER:

$0 + 2k\pi; \frac{\pi}{3} + 4k\pi; \frac{5\pi}{3} + 4k\pi$

Solve each equation for all values of θ if θ is measured in degrees.

15. $\cos 2\theta - \sin^2\theta + 2 = 0$

ANSWER:

$90° + k \cdot 180°$

16. $\sin^2\theta - \sin\theta = 0$

ANSWER:

$0° + k \cdot 180°,\ 90° + k \cdot 360°$

13-5 Solving Trigonometric Equations

17. $2\sin^2\theta - 1 = 0$

ANSWER:
$45° + k \cdot 90°$

18. $\cos\theta - 2\cos\theta\sin\theta = 0$

ANSWER:
$30° + k \cdot 360°, 150° + k \cdot 360°, 90° + k \cdot 180°$

19. $\cos 2\theta \sin\theta = 1$

ANSWER:
$270° + k \cdot 360°$

20. $\sin\theta\tan\theta - \tan\theta = 0$

ANSWER:
$0° + k \cdot 180°, 90° + k \cdot 360°$

21. **LIGHT** The number of hours of daylight d in Hartford, Connecticut, may be approximated by the equation $d = 3\sin\frac{2\pi}{365}t + 12$, where t is the number of days after March 21.
a. On what days will Hartford have exactly $10\frac{1}{2}$ hours of daylight?
b. Using the results in part **a**, tell what days of the year have at least $10\frac{1}{2}$ hours of daylight. Explain how you know.

ANSWER:

a. There will be $10\frac{1}{2}$ hours of daylight 213 and 335 days after March 21; that is, on October 20 and February 19.
b. Every day from February 19 to October 20; sample explanation: Since the longest day of the year occurs around June 22, the days between February 19 and October 20 must increase in length until June 22 and then decrease in length until October 20.

Solve each equation.

22. $\sin^2 2\theta + \cos^2\theta = 0$

ANSWER:
$\frac{\pi}{2} + \pi k$

23. $\tan^2\theta - 2\tan\theta + 1 = 0$

ANSWER:
$\frac{3\pi}{4} + \pi k$

24. $\cos^2\theta + 3\cos\theta = -2$

ANSWER:
$\pi + 2\pi k$

25. $\sin 2\theta - \cos\theta = 0$

ANSWER:
$\frac{\pi}{2} + \pi k, \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$

26. $\tan\theta = 1$

ANSWER:
$45° + k \cdot 180°$ *or* $\frac{\pi}{4} + k \cdot \pi$

27. $\cos 8\theta = 1$

ANSWER:
$0° + k \cdot 45°$ or $0° + k \cdot \frac{\pi}{4}$

28. $\sin\theta + 1 = \cos 2\theta$

ANSWER:
$0° + k \cdot 180°, 210° + k \cdot 360°, 330° + k \cdot 360°$

29. $2\cos^2\theta = \cos\theta$

ANSWER:
$\frac{\pi}{3} + 2k\pi, \frac{\pi}{2} + k\pi, \frac{5\pi}{3} + 2k\pi$

Solve each equation for the given interval.

30. $\cos^2\theta = \frac{1}{4};\ 0° \le \theta \le 360°$

ANSWER:
$60°, 120°, 240°, 300°$

31. $2\sin^2\theta = 1;\ 90° < \theta < 270°$

ANSWER:
$135°, 225°$

32. $\sin 2\theta - \cos\theta = 0;\ 0 \le \theta \le 2\pi$

ANSWER:

$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

33. $3\sin^2\theta = \cos^2\theta;\ 0 \le \theta \le \frac{\pi}{2}$

ANSWER:

$\frac{\pi}{6}$

34. $2\sin\theta + \sqrt{3} = 0;\ 180° < \theta < 360°$

ANSWER:

$240°, 300°$

35. $4\sin^2\theta - 1 = 0;\ 180° < \theta < 360°$

ANSWER:

$210°, 330°$

Solve each equation for all values of θ if θ is measured in radians

36. $\cos 2\theta + 3\cos\theta = 1$

ANSWER:

$\frac{\pi}{3} + 2k\pi,\ \frac{5\pi}{3} + 2k\pi$

37. $2\sin^2\theta = \cos\theta + 1$

ANSWER:

$\pi + 2k\pi,\ \frac{\pi}{3} + 2k\pi,\ \frac{5\pi}{3} + 2k\pi$

38. $\cos^2\theta - \frac{3}{2} = \frac{5}{2}\cos\theta$

ANSWER:

$\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$

39. $3\cos\theta - \cos\theta = 2$

ANSWER:

$0 + 2k\pi$

Solve each equation for all values of θ if θ is measured in degrees

40. $\sin\theta - \cos\theta = 0$

ANSWER:

$45° + k \cdot 180°$

41. $\tan\theta - \sin\theta = 0$

ANSWER:

$0° + k \cdot 180°$

42. $\sin^2\theta = 2\sin\theta + 3$

ANSWER:

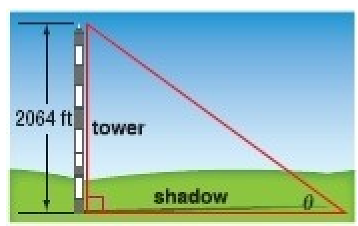
$270° + k \cdot 360°$

43. $4\sin^2 = 4\sin\theta - 1$

ANSWER:

$30° + k \cdot 360°, 150° + k \cdot 360°$

44. **ELECTRONICS** One of the tallest structures in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 2064 feet. What is the measure of θ if the length of the shadow is 1 mile?



ANSWER:

about 21°

Solve each equation.

45. $2\sin^2\theta = 3\sin\theta + 2$

ANSWER:

$\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°, 330° + k \cdot 360°$

46. $2\cos^2\theta + 3\sin\theta = 3$

ANSWER:

$\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi$ or

$30° + k \cdot 360°, 150° + k \cdot 360°, 90° + k \cdot 360°$

47. $\sin^2\theta + \cos 2\theta = \cos\theta$

ANSWER:

$0 + 2k\pi, \frac{\pi}{2} + k\pi$ or

$0° + k \cdot 360°, 90° + k \cdot 180°$

48. $2\cos^2\theta = -\cos\theta$

ANSWER:

$\frac{\pi}{2} + k\pi, \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$ *or*

$90° + k \cdot 180°, 120° + k \cdot 360°, 240° + k \cdot 360°$

49. **CCSS SENSE-MAKING** Due to ocean tides, the depth y in meters of the River Thames in London varies as a sine function of x, the hour of the day. On a certain day that function was

$y = 3\sin\left[\frac{\pi}{6}(x-4)\right] + 8$, where x = 0, 1, 2, ..., 24

corresponds to 12:00 midnight, 1:00 A.M., 2:00 A.M., ..., 12:00 midnight the next night.

a. What is the maximum depth of the River Thames on that day?

b. At what times does the maximum depth occur?

ANSWER:

a. 11m

b. 7:00 A.M. and 7:00 P.M.

Solve each equation if θ is measured in radians.

50. $(\cos\theta)(\sin 2\theta) - 2\sin\theta + 2 = 0$

ANSWER:

$\frac{\pi}{2} + 2\pi k$

51. $2\sin^2\theta + \left(\sqrt{2} - 1\right)\sin\theta = \frac{\sqrt{2}}{2}$

ANSWER:

$\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k, \frac{5\pi}{4} + 2\pi k, \frac{7\pi}{4} + 2\pi k$

Solve each equation if θ is measured in degrees.

52. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3}\sin\theta + \cos\theta$

ANSWER:

$30° + 360°k, 150° + 360°k, 330° + 360°k$

53. $1 - \sin^2\theta - \cos\theta = \frac{3}{4}$

ANSWER:

$120° + 360°k, 240° + 360°k$

Solve each equation.

54. $2\sin\theta = \sin 2\theta$

ANSWER:

πk

55. $\cos\theta\tan\theta - 2\cos^2\theta = -1$

ANSWER:

$\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$

56. **DIAMONDS** According to Snell's Law $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, *i* is the degree measure of the angle of incidence, and *r* is the degree measure of the angle of refraction.

a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?

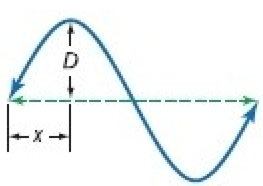
b. Explain how a gemologist might use Snell's Law to determine whether a diamond is genuine.

ANSWER:

a. 13.71°

b. Measure the angles of incidence and refraction to determine the index of refraction. If the index is 2.42, the diamond is genuine.

57. **CCSS PERSEVERANCE** A wave traveling in a guitar string can be modeled by the equation $D = 0.5\sin(6.5x)\sin(2500t)$, where *D* is the displacement in millimeters at the position *x* millimeters from the left end of the string at time *t* seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



ANSWER:

0.0026 second

58. **MULTIPLE REPRESENTATIONS** Consider the trigonometric inequality $\sin \theta \geq \frac{1}{2}$.

a. Tabular Construct a table of values for $0° \leq \theta \leq 360°$. For what values of θ is $\sin \theta \geq \frac{1}{2}$?

b. Graphical Graph $y = \sin \theta$ and $y = \frac{1}{2}$ on the same graph for $0° \leq \theta \leq 360°$. For what values of θ is the graph of $y = \sin \theta$ above the graph of $y = \frac{1}{2}$?

c. Analytic Based on your answers for parts *a* and *b*, solve $\sin \theta \geq \frac{1}{2}$ for all values of θ.

d. Algebraic Solve each inequality if $0 \leq \theta \leq 360°$. Then solve each for all values of θ.

i. $\cos \theta \geq \frac{\sqrt{2}}{2}$

ii. $2 \sin \theta \leq \sqrt{3}$

iii. $-\sin \theta \geq 0$

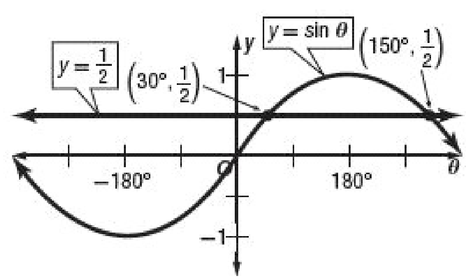
iv. $\cos \theta - 1 < -\frac{1}{2}$

ANSWER:

a.

θ	$\sin \theta$	θ	$\sin \theta$
0°	1	210°	$-\frac{1}{2}$
30°	$\frac{1}{2}$	225°	$-\frac{\sqrt{2}}{2}$
45°	$\frac{\sqrt{2}}{2}$	240°	$-\frac{\sqrt{3}}{2}$
60°	$\frac{\sqrt{3}}{2}$	270°	-1
90°	1	300°	$-\frac{\sqrt{3}}{2}$
120°	$\frac{\sqrt{3}}{2}$	315°	$-\frac{\sqrt{2}}{2}$
135°	$\frac{\sqrt{2}}{2}$	330°	$-\frac{1}{2}$
150°	$\frac{1}{2}$	360°	0
180°	0		

b. The graph of $y = \sin \theta$ is above the graph of $y = \frac{1}{2}$ for $30° < \theta < 150°$.



c. The graph of $y = \sin \theta$ is above the graph of $y = \frac{1}{2}$ for $30° < \theta < 150°$ and the period of $\sin \theta$ is [-360°, 360°], so the solutions of $\sin \theta > \frac{1}{2}$ are $30° + k \cdot 360° < \theta < 150° + k \cdot 360°$.

d.

i. $0° \leq \theta \leq 45°$ and $315° \leq \theta \leq 360°$; $0° + k \cdot 360° \leq \theta \leq 45° + k \cdot 360°$ and $315° + k \cdot 360° \leq \theta \leq 360° + k \cdot 360°$

ii. $0° \leq \theta \leq 60°$ and $120° \leq \theta \leq 360°$; $0° + k \cdot 360° \leq$

$\theta \leq 60° + k \cdot 360°$ and $120° + k \cdot 360° \leq \theta \leq 360° + k \cdot 360°$
iii. $180° \leq \theta \leq 360°$; $180° + k \cdot 360° \leq \theta \leq 360° + k \cdot 360°$
iv. $60° \leq \theta \leq 300°$; $60° + k \cdot 360° \leq \theta \leq 300° + k \cdot 360°$

59. **CHALLENGE** Solve $\sin 2x < \sin x$ for $0 \leq x \leq 2\pi$ without a calculator.

ANSWER:

$\frac{\pi}{3} < x < \pi$ *or* $\frac{5\pi}{3} < x < 2\pi$

60. **REASONING** Compare and contrast solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect?

ANSWER:

Each type of equation may require adding, subtracting, multiplying, or dividing each side by the same number. Quadratic and trigonometric equations can often be solved by factoring. Linear and quadratic equations do not require identities. All linear and quadratic equations can be solved algebraically, whereas some trigonometric equations may be graphed more easily by using a graphing calculator. A linear equation has at most one solution. A quadratic equation has at most two solutions. A trigonometric equation usually has infinitely many solutions, unless the values of the variable are restricted.

61. **WRITING IN MATH** Why do trigonometric equations often have infinitely many solutions?

ANSWER:

Sample answer: All trigonometric functions are periodic. Therefore, once one or more solutions are found for a certain interval, there will be additional solutions that can be found by adding integral multiples of the period of the function to those solutions.

62. **OPEN ENDED** Write an example of a trigonometric equation that has exactly two solutions if $0° \leq \theta \leq 360°$.

ANSWER:

Sample answer: $2\cos\theta = 0$, $90°$ and $270°$

63. **CHALLENGE** How many solutions in the interval $0° \leq \theta \leq 360°$ should you expect for $a\sin(b\theta + c) + d = d\left(\frac{a}{2}\right)$, if $a \neq 0$ and b is a positive integer?

ANSWER:

0, b, or $2b$ solutions

64. **EXTENDED RESPONSE** Charles received \$2500 for a graduation gift. He put it into a savings account in which the interest rate was 5.5% per year.
a. How much did he have in his savings account after 5 years if he made no deposits or withdrawals?
b. After how many years will the amount in his savings account have doubled?

ANSWER:

a. \$3267.40
b. about 13 years

65. **PROBABILITY** Find the probability of rolling three 3s if a number cube is rolled three times.

A. $\frac{1}{216}$

B. $\frac{1}{36}$

C. $\frac{1}{6}$

D. $\frac{1}{4}$

ANSWER:

A

66. Use synthetic substitution to find $f(-2)$ for the function below.
$f(x) = x^4 + 10x^2 + x + 8$
F. 62
G. 38
H. 30
J. 8

ANSWER:
F

67. **SAT/ACT** The pattern of dots below continues infinitely, with more dots being added at each step.

Which expression can be used to determine the number of dots in the *n*th step?
A $2n$
B $n(n + 2)$
C $n(n + 1)$
D $2(n + 2)$
E $2(n + 1)$

ANSWER:
E

Find the exact value of each expression.

68. cos 165°

ANSWER:
$-\frac{\sqrt{2+\sqrt{3}}}{2}$

69. $\sin 22\frac{1}{2}°$

ANSWER:
$\frac{\sqrt{2-\sqrt{2}}}{2}$

70. $\sin\frac{7\pi}{8}$

ANSWER:
$\frac{\sqrt{2-\sqrt{2}}}{2}$

71. $\cos\frac{7\pi}{12}$

ANSWER:
$-\frac{\sqrt{2-\sqrt{3}}}{2}$

Verify that each equation is an identity.

72. $\sin(270° - \theta) = -\cos\theta$

ANSWER:
$\sin(270° - \theta) \stackrel{?}{=} -\cos\theta$
$\sin 270° \cos\theta - \cos 270° \sin\theta \stackrel{?}{=} -\cos\theta$
$-1\cos\theta - 0 \stackrel{?}{=} -\cos\theta$
$-\cos\theta = -\cos\theta$ ✓

73. $\cos(90° + \theta) = -\sin\theta$

ANSWER:
$\cos(90° + \theta) \stackrel{?}{=} -\sin\theta$
$\cos 90° \cos\theta - \sin 90° \sin\theta \stackrel{?}{=} -\sin\theta$
$0 - 1\sin\theta - 0 \stackrel{?}{=} -\sin\theta$
$-\sin\theta = -\sin\theta$ ✓

74. $\cos(90° - \theta) = \sin\theta$

ANSWER:
$\cos(90° - \theta) \stackrel{?}{=} \sin\theta$
$\cos 90° \cos\theta + \sin 90° \sin\theta \stackrel{?}{=} \sin\theta$
$0 \cdot \cos\theta + 1 \cdot \sin\theta - 0 \stackrel{?}{=} \sin\theta$
$\sin\theta = \sin\theta$ ✓

75. $\sin(90° - \theta) = \cos\theta$

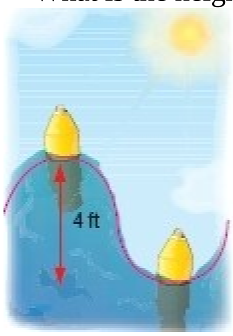
ANSWER:
$$\sin(90° - \theta) \stackrel{?}{=} \cos\theta$$
$$\sin 90° \cos\theta - \cos 90° \sin\theta \stackrel{?}{=} \cos\theta$$
$$1 \cdot \cos\theta - 0 \cdot \sin\theta \stackrel{?}{=} \cos\theta$$
$$\cos\theta - 0 \stackrel{?}{=} \cos\theta$$
$$\cos\theta = \cos\theta ✓$$

76. **WATER SAFETY** A harbor buoy bobs up and down with the waves. The distance between the highest and lowest points is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

a. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

b. Draw a graph showing the height of the buoy as a function of time.

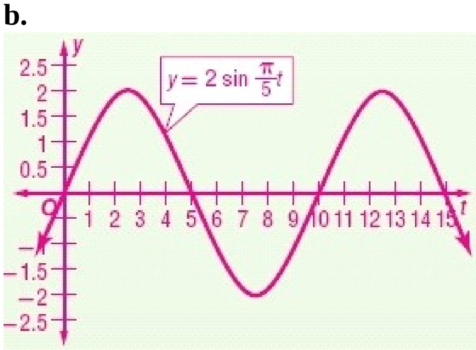
c. What is the height of the buoy after 12 seconds?



ANSWER:

a. $y = 2\sin\frac{\pi}{5}t$

b.



c. about 1.9 ft

Find the first three terms of each arithmetic series described.

77. $a_1 = 17, a_n = 197, S_n = 2247$

ANSWER:
17, 26, 35

78. $a_1 = -13, a_n = 427, S_n = 18{,}423$

ANSWER:
– 13, – 8, – 3

79. $n = 31, a_n = 78, S_n = 1023$

ANSWER:
– 12, – 9, – 6

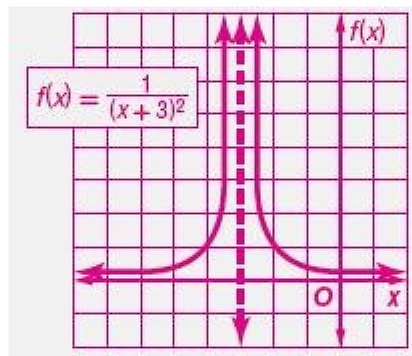
80. $n = 19, a_n = 103, S_n = 1102$

ANSWER:
13, 18, 23

Graph each rational function.

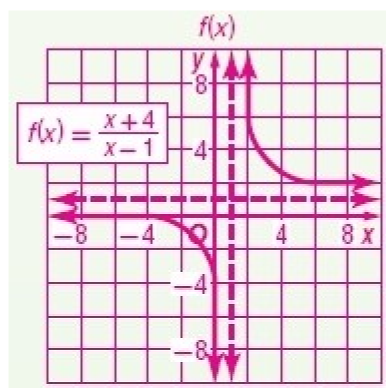
81. $f(x) = \dfrac{1}{(x+3)^2}$

ANSWER:



82. $f(x) = \dfrac{x+4}{x-1}$

ANSWER:



83. $f(x) = \dfrac{x+2}{x^2 - x - 6}$

ANSWER: